Chapter 1 Problem Solutions

S1.1

By definition, $\vec{A} = \nabla\phi = 8x\hat{i} + 3y^2\hat{j} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$, so

$\frac{\partial\phi}{\partial x} = 8x$ and $\frac{\partial\phi}{\partial y} = 3y^2$. Integrating these two expressions yields

$\phi = 4x^2 + c_1$ and $\phi = y^3 + c_2$. Adding these two expressions together gives

$\phi = 4x^2 + y^3 + c_1 + c_2$. Evaluating $\phi(1,1)$ and $\phi(0,1)$ yields

$\phi(1,1) = 4(1)^2 + (1)^3 + c_1 + c_2 = 5 + c_1 + c_2 = 8$ and

$\phi(0,1) = 4(0)^2 + (1)^3 + c_1 + c_2 = 1 + c_1 + c_2 = 4$. In both cases we find that

$c_1 + c_2 = 3$ leading to the conclusion that the functional expression for $\phi(x,y)$ is

$\phi(x,y) = 4x^2 + y^3 + 3$.

S1.2

(a) Since $\nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})\hat{i} + (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})\hat{j} + (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\hat{k}$

then, $\nabla\cdot\nabla\times\vec{V} = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})\cdot(\nabla\times\vec{V})$

$$= (\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 v}{\partial z\partial x}) + (\frac{\partial^2 u}{\partial y\partial z} - \frac{\partial^2 w}{\partial y\partial x}) + (\frac{\partial^2 v}{\partial z\partial x} - \frac{\partial^2 u}{\partial z\partial y}) = 0.$$

(b) Since $\vec{V}\cdot\nabla = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})$, then

$$(\vec{V}\cdot\nabla)\vec{V} = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})(u\hat{i} + v\hat{j} + w\hat{k})$$

$$
\begin{aligned}
&= u\frac{\partial u}{\partial x}\hat{i} + v\frac{\partial u}{\partial y}\hat{j} + w\frac{\partial u}{\partial z}\hat{k} \\
&+ u\frac{\partial v}{\partial x}\hat{i} + v\frac{\partial v}{\partial y}\hat{j} + w\frac{\partial u}{\partial z}\hat{k} \\
&+ u\frac{\partial w}{\partial x}\hat{i} + v\frac{\partial w}{\partial y}\hat{j} + w\frac{\partial w}{\partial z}\hat{k}
\end{aligned}
$$

This can be rewritten as

$$(\vec{V}\cdot\nabla)\vec{V} = (\vec{V}\cdot\nabla u)\hat{i} + (\vec{V}\cdot\nabla v)\hat{j} + (\vec{V}\cdot\nabla w)\hat{k}.$$

Now $\frac{1}{2}\nabla(\vec{V}\cdot\vec{V}) = \frac{1}{2}(\nabla u^2 + \nabla v^2 + \nabla w^2)$ and each of these three terms will take an expanded form such as

$$\frac{1}{2}(\nabla u^2) = \frac{1}{2}[2u\frac{\partial u}{\partial x}\hat{i} + 2u\frac{\partial u}{\partial y}\hat{j} + 2u\frac{\partial u}{\partial z}\hat{k}] = [u\frac{\partial u}{\partial x}\hat{i} + u\frac{\partial u}{\partial y}\hat{j} + u\frac{\partial u}{\partial z}\hat{k}]$$

so that

$$\frac{1}{2}\nabla(\vec{V}\cdot\vec{V}) = (u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x})\hat{i} + (u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y})\hat{j} + (u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z})\hat{k}.$$

Finally,

$$\vec{V}\times(\nabla\times\vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u & v & w \\ (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) & (\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) & (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) \end{vmatrix}$$

$$= [v(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) - w(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x})]\hat{i} + [w(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}) - u(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})]\hat{j} + [u(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}) - v(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z})]\hat{k}$$

which, when expanded in each component and added to the components of $(\vec{V}\cdot\nabla)\vec{V}$ yields an expression exactly equal to $\frac{1}{2}\nabla(\vec{V}\cdot\vec{V})$.

(c)

$$\begin{aligned}\nabla\cdot(f\vec{V}) &= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot(fu\hat{i}+fv\hat{j}+fw\hat{k}) \\ &= \frac{\partial}{\partial x}(fu)+\frac{\partial}{\partial y}(fv)+\frac{\partial}{\partial z}(fw) \\ &= u\frac{\partial f}{\partial x}+f\frac{\partial u}{\partial x}+v\frac{\partial f}{\partial y}+f\frac{\partial v}{\partial y}+w\frac{\partial f}{\partial z}+f\frac{\partial w}{\partial z} \\ &= \left(u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}\right)+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \\ &= \vec{V}\cdot\nabla f+f(\nabla\cdot\vec{V})\end{aligned}$$

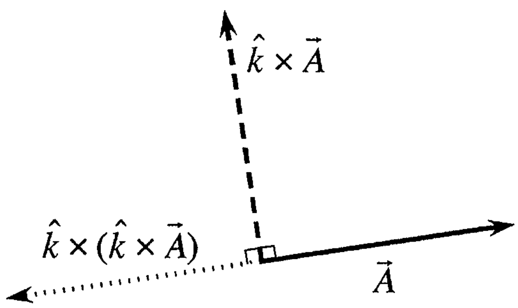
(d) We first calculate the vector (cross) product $\hat{k}\times\vec{A}$ as

$$\hat{k}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ A_1 & A_2 & 0\end{vmatrix}=-A_2\hat{i}+A_1\hat{j}.$$

Now, the cross product $k\times(\hat{k}\times\vec{A})$ is given by

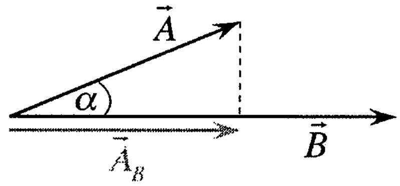
$$k\times(\hat{k}\times\vec{A})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ -A_2 & A_1 & 0\end{vmatrix}=-A_1\hat{i}-A_2\hat{j}=-\vec{A}.$$

(e) For any vector $\vec{A}$ (solid line), the dashed line represents $\hat{k}\times\vec{A}$ and the dotted line represents $k\times(\hat{k}\times\vec{A})$ making it clear that $k\times(\hat{k}\times\vec{A})=-\vec{A}$.



S1.3

We are after the component of $\vec{A}$ that is parallel to $\vec{B}$. From trigonometry, $|\vec{A}_B| = |\vec{A}|\cos\alpha$ where α is the angle between $\vec{A}$ and $\vec{B}$. We know that the dot product



formula is $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\alpha$ so that $\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = |\vec{A}|\cos\alpha = |\vec{A}_B|$. The direction of $\vec{A}_B$ is the same as the direction of $\vec{B}$ so that the final expression for $\vec{A}_B$ is

$$\vec{A}_B = \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|}\left(\frac{\vec{B}}{|\vec{B}|}\right).$$

S1.4

We can use the expressions for u and v given by (1.27a) and (1.27b) to answer this problem. Recall that for F_1=1 and ζ, D, and $F_2 = 0$ we get

$$u = \frac{1}{2}x \text{ and } v = -\frac{1}{2}y$$

and for F_2=1 with ζ, D, and $F_1 = 0$ we get

$$u = \frac{1}{2}y \text{ and } v = \frac{1}{2}x.$$

Thus, the combination F_1+F_2 gives

$$u = \frac{1}{2}(x + y) \text{ and } v = \frac{1}{2}(x - y).$$

Given these values for *u* and *v*, we simply evaluate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{2} + (-\frac{1}{2}) = 0$$

and

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{1}{2} - (\frac{1}{2}) = 0$$

proving that pure deformation has no vorticity or divergence in it.

S1.5

(a) Pure vorticity provides rotary motion as indicated in Fig. 1.1A. It seems apparent that a pure vortex can only *rotate* the bundle of isotherms depicted – it cannot make those isotherms come closer together or spread further apart. Thus, vorticity can only change the *direction* of the vector ∇T, it cannot change its magnitude.

(b) Pure convergence is characterized by a set of vectors, from every conceivable direction, acting inward toward a single point. Considering the subset of such vectors in Fig. 1.1A, it is possible to break each one into components across and along the isotherms. When one does this, one finds that there is no ability for convergence to *rotate* the isotherms, only to bring them closer together. So convergence can only change the *magnitude* of ∇T, not its direction.

(c) The deformation field is characterized by an axis of dilatation (along which the flow is stretched) and an axis of contraction (along which the flow is compressed). It is evident that the more perpendicular the isotherms are to the axis of contraction, the greater the ability of the deformation field to increase the magnitude of ∇T. If, however, the isotherms are fairly parallel to the axis of contraction, the deformation field will decrease the magnitude of ∇T.

Interestingly, only if the isotherms are exactly parallel to either axis will there be no rotation of the bundle of isotherms. Thus, for nearly every angle of orientation of isotherms to axes of dilatation/contraction the deformation field can also rotate ∇T. Intensification of $|\nabla T|$ will occur so long as the angle between the axis of dilatation and the isotherms is 45° or less. Rotation of ∇T will occur so long as that angle is neither 0° nor 90°. This property makes deformation extremely interesting in fluid dynamics.

S1.6

(a) If $A = \delta x \delta y$, then $\frac{dA}{dt}$ is given by

$$\frac{dA}{dt} = \frac{d}{dt}(\delta x \delta y) = \delta y \frac{d}{dt}(\delta x) + \delta x \frac{d}{dt}(\delta y)$$

by the chain rule. According to the hint given in the problem, this can be simplified by substituting $\frac{d}{dt}(\delta x) = \delta(\frac{dx}{dt}) = \delta u$ and $\frac{d}{dt}(\delta y) = \delta(\frac{dy}{dt}) = \delta v$ to yield

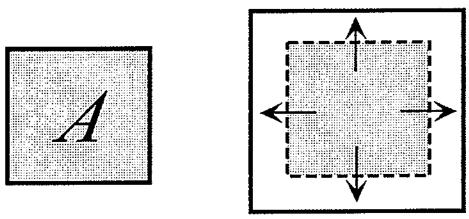
$$\frac{dA}{dt} = \delta u \delta y + \delta v \delta x.$$

(b) If we divide $\frac{dA}{dt}$ by A we get

$$\frac{1}{A}\frac{dA}{dt}=(\frac{1}{\delta x\delta y})\delta u\delta y+\delta v\delta x=\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}$$

which, in the limit of $\delta x, \delta y \to 0$, is the expression for divergence.

(c) If the area of the fluid element is increased, the flow must be divergent as only divergent flow can expand the area of the square as indicated below.



$t=0$ $t=t_1$

S1.7

The angle between the surfaces at the point is the angle between the *normals* to the surfaces at the point. A normal to $2x^2-y^2+z^2=9$ at (2,1,-2) is

$$\vec{A}=\nabla(2x^2-y^2+z^2)=4x\hat{i}-2y\hat{j}+2z\hat{k}=8\hat{i}-2\hat{j}-4\hat{k}.$$

A normal to $x^2-4y^2-3z=-5$ at (2,1,-2) is

$$\vec{B}=\nabla(x^2-4y^2-3z)=2x\hat{i}-8y\hat{j}-3\hat{k}=4\hat{i}-8\hat{j}-3\hat{k}.$$

Now we just take the dot product of $\vec{A}$ and $\vec{B}$, $\vec{A}\cdot\vec{B}=|\vec{A}||\vec{B}|\cos\alpha$, where α is the required angle. Thus,

$$(8\hat{i}-2\hat{j}-4\hat{k})\cdot(4\hat{i}-8\hat{j}-3\hat{k})=|8\hat{i}-2\hat{j}-4\hat{k}||4\hat{i}-8\hat{j}-3\hat{k}|\cos\alpha$$

which simplifies to $60 = (\sqrt{84})(\sqrt{89})\cos\alpha$. Therefore,

$$\alpha = \cos^{-1}\left(\frac{60}{(\sqrt{84})(\sqrt{89})}\right) = 46.06°$$

S1.8

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = 2xyz^2\hat{i} + x^2z^2\hat{j} + 2x^2yz\hat{k}$$

Solving each of these three separate derivative expressions for $\phi(x,y,z)$ yields;

$$\frac{\partial\phi}{\partial x} = 2xyz^2 \quad \text{so} \quad \phi(x,y,z) = x^2yz^2 + C_x$$

$$\frac{\partial\phi}{\partial y} = x^2z^2 \quad \text{so} \quad \phi(x,y,z) = x^2yz^2 + C_y$$

$$\frac{\partial\phi}{\partial z} = 2x^2yz \quad \text{so} \quad \phi(x,y,z) = x^2yz^2 + C_z.$$

Thus, $\phi(x,y,z) = x^2yz^2 + K$ where $K = C_x + C_y + C_z$. We solve for K by evaluating $\phi(x,y,z)$ at the indicated point.

Since $\phi(1,-2,2) = (1)^2(-2)(2)^2 + K = 4$, we find that $K = 12$ so that the final answer is $\phi(x,y,z) = x^2yz^2 + 12$.

S1.9

$$\nabla^2(\alpha\beta) = \nabla\cdot\nabla(\alpha\beta) = \nabla\cdot[(\beta\frac{\partial\alpha}{\partial x} + \alpha\frac{\partial\beta}{\partial x})\hat{i} + (\beta\frac{\partial\alpha}{\partial y} + \alpha\frac{\partial\beta}{\partial y})\hat{j}]$$

$$= \frac{\partial}{\partial x}(\beta\frac{\partial\alpha}{\partial x} + \alpha\frac{\partial\beta}{\partial x}) + \frac{\partial}{\partial y}(\beta\frac{\partial\alpha}{\partial y} + \alpha\frac{\partial\beta}{\partial y})$$